EXERCISES

1. The Main Theorem of Galois Theory

1. Determine the irreducible polynomial for $i + \sqrt{2}$ over $\mathbb{Q}$.
2. Prove that the set $(1, i, \sqrt{2}, i\sqrt{2})$ is a basis for $\mathbb{Q}(i, \sqrt{2})$ over $\mathbb{Q}$.
3. Determine the intermediate fields between $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{2}, \sqrt{3})$.
4. Determine the intermediate fields of an arbitrary biquadratic extension without appealing to the Main Theorem.
5. Prove that the automorphism $\mathbb{Q}(\sqrt{2})$ sending $\sqrt{2}$ to $-\sqrt{2}$ is discontinuous.
6. Determine the degree of the splitting field of the following polynomials over $\mathbb{Q}$.
   (a) $x^4 - 1$ **(b)** $x^3 - 2$ **(c)** $x^4 + 1$
7. Let α denote the positive real fourth root of 2. Factor the polynomial $x^4 - 2$ into irreducible factors over each of the fields $\mathbb{Q}$, $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{2}, i)$, $\mathbb{Q}(\alpha)$, $\mathbb{Q}(\alpha, i)$.
8. Let $\zeta = e^{2\pi i/5}$.
   (a) Prove that $K = \mathbb{Q}(\zeta)$ is a splitting field for the polynomial $x^5 - 1$ over $\mathbb{Q}$, and determine the degree $[K : \mathbb{Q}]$.
   (b) Without using Theorem (1.11), prove that K is a Galois extension of $\mathbb{Q}$, and determine its Galois group.
9. Let K be a quadratic extension of the form $F(\alpha)$, where $\alpha^2 = a \in F$. Determine all elements of K whose squares are in F.
10. Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$. Determine $[K : \mathbb{Q}]$, prove that K is a Galois extension of $\mathbb{Q}$, and determine its Galois group.
11. Let K be the splitting field over $\mathbb{Q}$ of the polynomial $f(x) = (x^2 - 2x - 1)(x^2 - 2x - 7)$. Determine $G(K/\mathbb{Q})$, and determine all intermediate fields explicitly.
12. Determine all automorphisms of the field $\mathbb{Q}(\sqrt[3]{2})$.
13. Let K/F be a finite extension. Prove that the Galois group $G(K/F)$ is a finite group.
14. Determine all the quadratic number fields $\mathbb{Q}[\sqrt{d}]$ which contain a primitive pth root of unity, for some prime $p \neq 2$.
15. Prove that every Galois extension K/F whose Galois group is the Klein four group is biquadratic.
16. Prove or disprove: Let $f(x)$ be an irreducible cubic polynomial in $\mathbb{Q}[x]$ with one real root α. The other roots form a complex conjugate pair $\beta, \bar{\beta}$, so the field $L = \mathbb{Q}(\beta)$ has an automorphism σ which interchanges $\beta, \bar{\beta}$.
17. Let K be a Galois extension of a field F such that $G(K/F) \approx C_2 \times C_{12}$. How many intermediate fields L are there such that **(a)** $[L : F] = 4$, **(b)** $[L : F] = 9$, **(c)** $G(K/L) \approx C_4$?
18. Let $f(x) = x^4 + bx^2 + c \in F[x]$, and let K be the splitting field of f. Prove that $G(K/F)$ is contained in a dihedral group D_4.
19. Let $F = \mathbb{F}_2(u)$ be the rational function field over the field of two elements. Prove that the polynomial $x^2 - u$ is irreducible in $F[x]$ and that it has two equal roots in a splitting field.